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Some identities of Korobov-type polynomials associated with p-adic integrals on $\mathbb{Z}_p$

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Abstract
In this paper, we consider Korobov-type polynomials derived from the bosonic and fermionic p-adic integrals on $\mathbb{Z}_p$, and we give some interesting and new identities of those polynomials and of their mixed-types.

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1 Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the bosonic p-adic integral on $\mathbb{Z}_p$ is defined by

$$\begin{aligned} I_0(f) &= \int_{\mathbb{Z}_p} f(x)\,d\mu_0(x) \\ &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_0\left(x + p^N\mathbb{Z}_p\right) \\ &= \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x). \end{aligned} \tag{1.1}$$

Thus, by (1.1), we get

$$I_0(f_1) = I_0(f) + f'(0), \quad \text{where } f_1(x) = f(x+1) \text{ (see [1–4])}. \tag{1.2}$$

The fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim as

$$\begin{aligned} I_{-1}(f) &= \int_{\mathbb{Z}_p} f(x)\,d\mu_{-1}(x) \\ &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_{-1}\bigl(x+p^N\mathbb{Z}_p\bigr) \\ &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x. \end{aligned} \tag{1.3}$$

Thus, from (1.3), we have

$$I_{-1}(f_1) = -I_{-1}(f) + 2f(0) \quad \text{(see [1])}. \tag{1.4}$$

From (1.2) and (1.4), we can derive the following equations:

$$I_0(f_n) - I_0(f) = \sum_{l=0}^{n-1} f'(l), \qquad I_{-1}(f_n) + (-1)^{n-1}I_{-1}(f) = 2\sum_{l=0}^{n-1} f(l), \tag{1.5}$$

where $f_n(x) = f(x+n)$, $f'(l) = \frac{df(x)}{dx}|_{x=l}$ (see [1, 4, 5]).

As is well known, the Bernoulli polynomials of order r ($\in \mathbb{N}$) are defined by the generating function

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!} \quad \text{(see [1, 6])}. \tag{1.6}$$

When $x = 0$, $B_n^{(r)} = B_n^{(r)}(0)$ are called the Bernoulli numbers of order r. In particular, if $r = 1$, $B_n(x) = B_n^{(1)}(x)$ are called the ordinary Bernoulli polynomials.

The Euler polynomials of order r are also given by the generating function

$$\left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!} \quad \text{(see [1, 6])}. \tag{1.7}$$

When $x = 0$, $E_n^{(r)} = E_n^{(r)}(0)$ are called the Euler numbers of order r. In particular, if $r = 1$, then $E_n(x) = E_n^{(1)}(x)$ are called the ordinary Euler polynomials.

The Daehee polynomials of order r are defined by the generating function

$$\left(\frac{\log(1+t)}{t}\right)^r (1+t)^x = \sum_{n=0}^{\infty} D_n^{(r)}(x)\frac{t^n}{n!} \quad \text{(see [7])}. \tag{1.8}$$

When $x = 0$, $D_n^{(r)} = D_n^{(r)}(0)$ are called the Daehee numbers of order r. In particular, if $r = 1$, then $D_n(x) = D_n^{(1)}(x)$ are called the ordinary Daehee polynomials. Now, we introduce the Changhee polynomials of order r given by the generating function

$$\left(\frac{2}{t+2}\right)^r (1+t)^x = \sum_{n=0}^{\infty} Ch_n^{(r)}(x)\frac{t^n}{n!} \quad \text{(see [6])}. \tag{1.9}$$

When $x = 0$, $Ch_n^{(r)} = Ch_n^{(r)}(0)$ are called the Changhee numbers of order r. In particular, if $r = 1$, then $Ch_n(x) = Ch_n^{(1)}(x)$ are called the ordinary Changhee polynomials.

Recently, Korobov introduced the special polynomials given by the generating function

$$\frac{\lambda t}{(t+1)^{\lambda}-1}(1+t)^x = \sum_{n=0}^{\infty} K_n(x \mid \lambda)\frac{t^n}{n!} \quad (\lambda \in \mathbb{N}) \text{ (see [8–12])}. \tag{1.10}$$

Note that $\lim_{\lambda\to 0} K_n(x \mid \lambda) = b_n(x)$, where $b_n(x)$ are the Bernoulli polynomials of the second kind defined by the generating function

$$\left(\frac{t}{\log(1+t)}\right)(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!} \quad \text{(see [7, 13])}. \tag{1.11}$$

In this paper, we define the higher-order Korobov polynomials given by the generating function

$$\left(\frac{\lambda t}{(t+1)^{\lambda}-1}\right)^r (1+t)^x = \sum_{n=0}^{\infty} K_n^{(r)}(x \mid \lambda)\frac{t^n}{n!}. \tag{1.12}$$

When $x = 0$, $K_n^{(r)}(\lambda) = K_n^{(r)}(0 \mid \lambda)$ are called the Korobov numbers of order r. In particular, if $r = 1$, then $K_n(\lambda) = K_n^{(1)}(0 \mid \lambda) = K_n(0 \mid \lambda)$ are called the ordinary Korobov numbers. Now, we consider the Korobov-type Changhee polynomials which are called the λ-Changhee polynomials as follows:

$$\left(\frac{2}{(1+t)^{\lambda}+1}\right)(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x \mid \lambda)\frac{t^n}{n!}. \tag{1.13}$$

When $x = 0$, $Ch_n(\lambda) = Ch_n(0 \mid \lambda)$ are called λ-Changhee numbers. Note that $\lim_{\lambda\to 1} Ch_n(x \mid \lambda) = Ch_n(x)$, $\lim_{\lambda\to 0} Ch_n(x \mid \lambda) = (x)_n$, where

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l \quad \text{(see [7])}.$$

For $r \in \mathbb{N}$, the λ-Changhee polynomials of order r are defined by the generating function

$$\left(\frac{2}{(1+t)^{\lambda}+1}\right)^r (1+t)^x = \sum_{n=0}^{\infty} Ch_n^{(r)}(x \mid \lambda)\frac{t^n}{n!}. \tag{1.14}$$

The Stirling numbers of the second kind are defined by the generating function

$$\left(e^t-1\right)^n = n!\sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!} \quad \text{(see [7, 13])}. \tag{1.15}$$

The Korobov polynomials (of the first kind) were introduced in [10] as the degenerate version of the Bernoulli polynomials of the second kind. In recent years, many researchers studied various kinds of degenerate versions of some familiar polynomials like

Bernoulli polynomials, Euler polynomials and their variants by means of generating functions, p-adic integrals and umbral calculus (see [1, 6, 12, 14]).

Here in this paper we introduce two Korobov-type polynomials obtained from the same function, namely the one by performing bosonic p-adic integrals on $\mathbb{Z}_p$ and the other by carrying out fermionic p-adic integrals on $\mathbb{Z}_p$. In addition, we consider their higher-order versions and some mixed-types of them by considering multivariate p-adic integrals. In conclusion, we will obtain some connections between these new polynomials and Bernoulli polynomials, Euler polynomials, Daehee numbers and Bernoulli numbers of the second kind.

2 Korobov-type polynomials

For $\lambda \in \mathbb{N}$, by (1.2), we get

$$\begin{aligned}\int_{\mathbb{Z}_p} (1+t)^{\lambda y+x}\, d\mu_0(y) &= \frac{\lambda \log(1+t)}{(1+t)^{\lambda}-1}(1+t)^x \\ &= \left(\frac{\lambda t}{(1+t)^{\lambda}-1}\right)\left(\frac{\log(1+t)}{t}\right)(1+t)^x \\ &= \left(\sum_{l=0}^{\infty} K_l(x\mid\lambda)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} D_m \frac{t^m}{m!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n} K_l(x\mid\lambda) D_{n-l}\frac{n!}{l!(n-l)!}\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} K_l(x\mid\lambda) D_{n-l}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.1}$$

From (2.1), we have

$$\int_{\mathbb{Z}_p}\binom{\lambda y+x}{n} d\mu_0(y) = \frac{1}{n!}\sum_{l=0}^{n}\binom{n}{l} K_l(x\mid\lambda) D_{n-l}. \tag{2.2}$$

Therefore, by (2.2), we obtain the following theorem.

Theorem 2.1 *For $n \geq 0$, we have*

$$\int_{\mathbb{Z}_p}\binom{\lambda y+x}{n} d\mu_0(y) = \frac{1}{n!}\sum_{l=0}^{n}\binom{n}{l} K_l(x\mid\lambda) D_{n-l}.$$

Now, we observe that

$$\begin{aligned}\int_{\mathbb{Z}_p} (\lambda y+x)_n\, d\mu_0(y) &= \sum_{l=0}^{n} S_1(n,l)\int_{\mathbb{Z}_p}(\lambda y+x)^l\, d\mu_0(y) \\ &= \sum_{l=0}^{n} S_1(n,l)\lambda^l \int_{\mathbb{Z}_p}\left(y+\frac{x}{\lambda}\right)^l d\mu_0(y) \\ &= \sum_{l=0}^{n} S_1(n,l)\lambda^l B_l\left(\frac{x}{\lambda}\right).\end{aligned} \tag{2.3}$$

Therefore by (2.3), we obtain the following corollary.

Corollary 2.2 *For $n \geq 0$, we have*

$$\sum_{l=0}^{n} S_1(n,l)\lambda^l B_l\left(\frac{x}{\lambda}\right) = \sum_{l=0}^{n} \binom{n}{l} K_l(x \mid \lambda) D_{n-l}.$$

From (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} K_n(x \mid \lambda)\frac{t^n}{n!} &= \frac{\lambda t}{(1+t)^{\lambda}-1}(1+t)^x \\ &= \left(\frac{t}{\log(1+t)}\right)\int_{\mathbb{Z}_p} (1+t)^{\lambda y+x}\, d\mu_0(y) \\ &= \left(\sum_{l=0}^{\infty} b_l \frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\int_{\mathbb{Z}_p} (\lambda y+x)_m\, d\mu_0(y)\frac{t^m}{m!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m} b_{n-m}\sum_{l=0}^{m} S_1(m,l)\lambda^l B_l\left(\frac{x}{\lambda}\right)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.4}$$

Therefore, by (2.4), we obtain the following corollary.

Corollary 2.3 *For $n \geq 0$, we have*

$$K_n(x \mid \lambda) = \sum_{m=0}^{n}\binom{n}{m} b_{n-m}\sum_{l=0}^{m} S_1(m,l)\lambda^l B_l\left(\frac{x}{\lambda}\right).$$

By replacing t by $e^t - 1$ in (1.10), we get

$$\begin{aligned}\sum_{m=0}^{\infty} K_m(x \mid \lambda)\frac{(e^t-1)^m}{m!} &= \frac{\lambda(e^t-1)}{e^{\lambda t}-1}e^{xt} \\ &= \frac{\lambda t}{e^{\lambda t}-1}e^{(\frac{x}{\lambda})\lambda t}\frac{e^t-1}{t} \\ &= \left(\sum_{m=0}^{\infty}\lambda^m B_m\left(\frac{x}{\lambda}\right)\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}\frac{1}{l+1}\frac{t^l}{l!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\lambda^{n-l}B_{n-l}\left(\frac{x}{\lambda}\right)\frac{1}{l+1}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.5}$$

On the other hand,

$$\begin{aligned}\sum_{m=0}^{\infty} K_m(x \mid \lambda)\frac{1}{m!}\left(e^t-1\right)^m &= \sum_{m=0}^{\infty} K_m(x \mid \lambda)\sum_{l=m}^{\infty} S_2(l,m)\frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} K_m(x \mid \lambda) S_2(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.6}$$

Therefore, by (2.5) and (2.6), we obtain the following theorem.

Theorem 2.4 *For $n \geq 0$, we have*

$$\sum_{l=0}^{n} \binom{n}{l} \lambda^{n-l} B_{n-l}\left(\frac{x}{\lambda}\right) \frac{1}{l+1} = \sum_{m=0}^{n} K_m(x \mid \lambda) S_2(n,m).$$

It is easy to show that

$$\int_{\mathbb{Z}_p} f(x)\,d\mu_0(x) = \frac{1}{d} \sum_{a=0}^{d-1} \int_{\mathbb{Z}_p} f(a+dx)\,d\mu_0(x) \quad (d \in \mathbb{N}). \tag{2.7}$$

By (2.7), we get

$$\begin{aligned} \int_{\mathbb{Z}_p} (1+t)^{\lambda x}\,d\mu_0(x) &= \frac{1}{d} \sum_{a=0}^{d-1} \int_{\mathbb{Z}_p} (1+t)^{(a+dx)\lambda}\,d\mu_0(x) \\ &= \frac{1}{d} \sum_{a=0}^{d-1} (1+t)^{a\lambda} \int_{\mathbb{Z}_p} (1+t)^{\lambda dx}\,d\mu_0(x) \\ &= \frac{1}{d} \sum_{a=0}^{d-1} (1+t)^{a\lambda} \frac{\lambda d \log(1+t)}{(1+t)^{d\lambda}-1}. \end{aligned} \tag{2.8}$$

On the other hand,

$$\int_{\mathbb{Z}_p} (1+t)^{\lambda x}\,d\mu_0(x) = \frac{\lambda \log(1+t)}{(1+t)^{\lambda}-1}. \tag{2.9}$$

Thus, by (2.8) and (2.9), we get

$$\frac{\lambda t}{(1+t)^{\lambda}-1}(1+t)^{x} = \frac{1}{d} \sum_{a=0}^{d-1} \frac{\lambda d t}{(1+t)^{\lambda d}-1}(1+t)^{a\lambda+x}. \tag{2.10}$$

Therefore, by (1.10) and (2.10), we obtain the following theorem.

Theorem 2.5 *For $n \geq 0$ and $d \in \mathbb{N}$, we have*

$$K_n(x \mid \lambda) = \frac{1}{d} \sum_{a=0}^{d-1} K_n(a\lambda + x \mid \lambda d).$$

From (1.5), we can derive the following equation:

$$\int_{\mathbb{Z}_p} (1+t)^{(x+n)\lambda}\,d\mu_0(x) - \int_{\mathbb{Z}_p} (1+t)^{\lambda x}\,d\mu_0(x) = \lambda \log(1+t) \sum_{l=0}^{n-1} (1+t)^{\lambda l} \quad (n \in \mathbb{N}). \tag{2.11}$$

Thus, by (2.11), we get

$$\int_{\mathbb{Z}_p} (1+t)^{\lambda x}\,d\mu_0(x) = \frac{\lambda \log(1+t)}{(1+t)^{\lambda n}-1} \sum_{l=0}^{n-1} (1+t)^{\lambda l}. \tag{2.12}$$

From (2.12), we have

$$\frac{t}{\log(1+t)}\int_{\mathbb{Z}_p}(1+t)^{\lambda x}\,d\mu_0(x)=\frac{1}{n}\sum_{l=0}^{n-1}\frac{\lambda nt}{(1+t)^{\lambda n}-1}(1+t)^{\lambda l}$$
$$=\sum_{m=0}^{\infty}\left(\frac{1}{n}\sum_{l=0}^{n-1}K_m(\lambda l\mid\lambda n)\right)\frac{t^m}{m!}. \tag{2.13}$$

On the other hand,

$$\frac{t}{\log(1+t)}\int_{\mathbb{Z}_p}(1+t)^{\lambda x}\,d\mu_0(x)=\left(\sum_{n=0}^{\infty}b_n\frac{t^n}{n!}\right)\left(\sum_{l=0}^{\infty}\int_{\mathbb{Z}_p}(\lambda x)_l\,d\mu_0(x)\frac{t^l}{l!}\right)$$
$$=\sum_{m=0}^{\infty}\left(\sum_{l=0}^{m}\binom{m}{l}b_{m-l}\int_{\mathbb{Z}_p}(\lambda x)_l\,d\mu_0(x)\right)\frac{t^m}{m!}$$
$$=\sum_{m=0}^{\infty}\left(\sum_{l=0}^{m}\binom{m}{l}b_{m-l}\sum_{n=0}^{l}S_1(l,n)\lambda^nB_n\right)\frac{t^m}{m!}. \tag{2.14}$$

Therefore, by (2.13) and (2.14), we obtain the following theorem.

Theorem 2.6 *For $n\in\mathbb{N}$, $m\geq 0$, we have*

$$\frac{1}{n}\sum_{l=0}^{n-1}K_m(\lambda l\mid\lambda n)=\sum_{l=0}^{m}\binom{m}{l}b_{m-l}\sum_{n=0}^{l}S_1(l,n)\lambda^nB_n.$$

Remark By (1.5), we easily get

$$\int_{\mathbb{Z}_p}\bigl(\lambda(x+n)\bigr)_m\,d\mu_0(x)-\int_{\mathbb{Z}_p}(\lambda x)_m\,d\mu_0(x)=\sum_{l=0}^{n-1}\sum_{k=1}^{m}kS_1(m,k)\lambda^kl^{k-1}. \tag{2.15}$$

Hence, by Theorem 2.1 and (2.15), we see

$$\sum_{l=0}^{m}K_l(\lambda n\mid\lambda)D_{m-l}\binom{m}{l}-\sum_{l=0}^{m}K_l(\lambda)D_{m-l}\binom{m}{l}=\sum_{l=0}^{n-1}\sum_{k=1}^{m}kS_1(m,k)\lambda^kl^{k-1}.$$

Now, we consider the multivariate p-adic integral on $\mathbb{Z}_p$ given by

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{\lambda(x_1+\cdots+x_r)+x}\,d\mu_0(x_1)\cdots d\mu_0(x_r). \tag{2.16}$$

By (1.2) and (2.16), we get

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{\lambda(x_1+\cdots+x_r)+x}\,d\mu_0(x_1)\cdots d\mu_0(x_r)$$
$$=\left(\frac{\lambda\log(1+t)}{(1+t)^{\lambda}-1}\right)^r(1+t)^x$$
$$=\left(\frac{\lambda t}{(1+t)^{\lambda}-1}\right)^r(1+t)^x\left(\frac{\log(1+t)}{t}\right)^r. \tag{2.17}$$

Thus, by (2.17), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} K_n^{(r)}(x \mid \lambda)\frac{t^n}{n!} \\
&= \left(\frac{\lambda t}{(1+t)^{\lambda}-1}\right)^r (1+t)^x \\
&= \left(\frac{t}{\log(1+t)}\right)^r \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{\lambda(x_1+\cdots+x_r)+x}\, d\mu_0(x_1)\cdots d\mu_0(x_r) \\
&= \left(\sum_{m=0}^{\infty} b_m^{(r)}\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \big(\lambda(x_1+\cdots+x_r)+x\big)_l\, d\mu_0(x_1)\cdots d\mu_0(x_r)\frac{t^l}{l!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{k=0}^{l} \lambda^k S_1(l,k) B_k^{(r)}\left(\frac{x}{\lambda}\right)\binom{n}{l} b_{n-l}^{(r)}\right)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients on both sides, we obtain the following theorem.

Theorem 2.7 *For $n \geq 0$, we have*

$$K_n^{(r)}(x \mid \lambda) = \sum_{l=0}^{n}\sum_{k=0}^{l} \lambda^k S_1(l,k) B_k^{(r)}\left(\frac{x}{\lambda}\right)\binom{n}{l} b_{n-l}^{(r)}.$$

By replacing t by $e^t - 1$ in (1.12), we get

$$\begin{aligned}
\left(\frac{\lambda(e^t-1)}{e^{\lambda t}-1}\right)^r e^{xt} &= \sum_{m=0}^{\infty} K_m^{(r)}(x \mid \lambda)\frac{1}{m!}\left(e^t-1\right)^m \\
&= \sum_{m=0}^{\infty} K_m^{(r)}(x \mid \lambda)\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} K_m^{(r)}(x \mid \lambda) S_2(n,m)\right)\frac{t^n}{n!}. \qquad (2.18)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\left(\frac{\lambda(e^t-1)}{e^{\lambda t}-1}\right)^r e^{xt} &= \left(\frac{\lambda t}{e^{\lambda t}-1}\right)^r e^{(\frac{x}{\lambda})\lambda t}\left(\frac{e^t-1}{t}\right)^r \\
&= \left(\sum_{m=0}^{\infty} \lambda^m B_m^{(r)}\left(\frac{x}{\lambda}\right)\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty} r! S_2(l+r,r)\frac{t^l}{(l+r)!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n} \frac{\binom{n}{l}}{\binom{l+r}{r}} \lambda^{n-l} S_2(l+r,r) B_{n-l}^{(r)}\left(\frac{x}{\lambda}\right)\right)\frac{t^n}{n!}. \qquad (2.19)
\end{aligned}$$

Therefore, by (2.18) and (2.19), we obtain the following theorem.

Theorem 2.8 *For $n \geq 0$, we have*

$$\sum_{m=0}^{n} K_m^{(r)}(x \mid \lambda) S_2(n,m) = \sum_{l=0}^{n} \frac{\binom{n}{l}}{\binom{l+r}{r}} \lambda^{n-l} S_2(l+r,r) B_{n-l}^{(r)}\left(\frac{x}{\lambda}\right).$$

From (1.3), we can derive the following equation:

$$\int_{\mathbb{Z}_p} (1+t)^{\lambda y+x}\, d\mu_{-1}(y) = \frac{2}{(1+t)^\lambda+1}(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x \mid \lambda)\frac{t^n}{n!}. \tag{2.20}$$

Thus, by (2.20), we get

$$\int_{\mathbb{Z}_p} \binom{\lambda y+x}{n}\, d\mu_{-1}(y) = \frac{1}{n!}Ch_n(x \mid \lambda) \quad (n \geq 0). \tag{2.21}$$

We observe that

$$\begin{aligned}
\sum_{n=0}^{\infty} Ch_n(x \mid \lambda)\frac{t^n}{n!} &= \left(\frac{2}{(1+t)^\lambda+1}\right)(1+t)^x \\
&= \left(\sum_{l=0}^{\infty} Ch_l(\lambda)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(x)_m\frac{t^m}{m!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}(x)_m Ch_{n-m}(x)\binom{n}{m}\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.22}$$

Therefore, by (2.21) and (2.22), we obtain the following theorem.

Theorem 2.9 *For $n \geq 0$, we have*

$$\int_{\mathbb{Z}_p} \binom{\lambda y+x}{n}\, d\mu_{-1}(y) = \frac{1}{n!}Ch_n(x \mid \lambda) = \frac{1}{n!}\sum_{m=0}^{n}(x)_m Ch_{n-m}(\lambda)\binom{n}{m}.$$

From (2.20), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} Ch_n(x \mid \lambda)\frac{t^n}{n!} &= \int_{\mathbb{Z}_p} (1+t)^{\lambda y+x}\, d\mu_{-1}(y) \\
&= \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}(\lambda y+x)_n\, d\mu_{-1}(y)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n} S_1(n,l)\lambda^l\int_{\mathbb{Z}_p}\left(y+\frac{x}{\lambda}\right)^l d\mu_{-1}(y)\right)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n} S_1(n,l)\lambda^l E_l\left(\frac{x}{\lambda}\right)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.23}$$

By replacing t by e^t-1 in (1.13), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} Ch_n(x \mid \lambda)\frac{1}{n!}\left(e^t-1\right)^n &= \frac{2}{e^{\lambda t}+1}e^{xt} \\
&= \frac{2}{e^{\lambda t}+1}e^{(\frac{x}{\lambda})\lambda t} \\
&= \sum_{n=0}^{\infty} E_n\left(\frac{x}{\lambda}\right)\lambda^n\frac{t^n}{n!}.
\end{aligned} \tag{2.24}$$

On the other hand,

$$\begin{aligned}\sum_{n=0}^{\infty} Ch_n(x \mid \lambda)\frac{1}{n!}\left(e^t-1\right)^n &= \sum_{m=0}^{\infty} Ch_m(x \mid \lambda)\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} Ch_m(x \mid \lambda)S_2(n,m)\right)\frac{t^n}{n!}. \qquad (2.25)\end{aligned}$$

Therefore, by (2.23), (2.24), and (2.25), we obtain the following theorem.

Theorem 2.10 *For $n \geq 0$, we have*

$$E_n\left(\frac{x}{\lambda}\right) = \lambda^{-n}\sum_{m=0}^{n} Ch_m(x \mid \lambda)S_2(n,m)$$

and

$$Ch_n(x \mid \lambda) = \sum_{l=0}^{n} S_1(n,l)\lambda^l E_l\left(\frac{x}{\lambda}\right).$$

By replacing t by $e^t - 1$ in (1.14), we get

$$E_n^{(r)}\left(\frac{x}{\lambda}\right) = \lambda^{-n}\sum_{m=0}^{n} Ch_m^{(r)}(x \mid \lambda)S_2(n,m). \qquad (2.26)$$

From (1.14), we can derive the following equation:

$$\begin{aligned}\sum_{n=0}^{\infty} Ch_n^{(r)}(x \mid \lambda)\frac{t^n}{n!} &= \left(\frac{2}{(1+t)^{\lambda}+1}\right)^r (1+t)^x \\ &= \left(\frac{2}{e^{\lambda\log(1+t)}+1}\right)^r e^{x\log(1+t)} \\ &= \sum_{n=0}^{\infty} E_n^{(r)}\left(\frac{x}{\lambda}\right)\frac{1}{n!}\lambda^n\left(\log(1+t)\right)^n \\ &= \sum_{m=0}^{\infty} E_m^{(r)}\left(\frac{x}{\lambda}\right)\lambda^m\sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} E_m^{(r)}\left(\frac{x}{\lambda}\right)\lambda^m S_1(n,m)\right)\frac{t^n}{n!}. \qquad (2.27)\end{aligned}$$

Therefore, by (2.26) and (2.27), we obtain the following theorem.

Theorem 2.11 *For $n \geq 0$, we have*

$$Ch_n^{(r)}(x \mid \lambda) = \sum_{m=0}^{n} E_m^{(r)}\left(\frac{x}{\lambda}\right)\lambda^m S_1(n,m)$$

and

$$E_n^{(r)}\left(\frac{x}{\lambda}\right) = \frac{1}{\lambda^n}\sum_{m=0}^{n} Ch_m^{(r)}(x \mid \lambda)S_2(n,m).$$

Let us observe the following multivariate fermionic p-adic integral on $\mathbb{Z}_p$:

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{\lambda(x_1+\cdots+x_r)+x} \, d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
&\quad = \left(\frac{2}{(1+t)^{\lambda}+1} \right)^r (1+t)^x \\
&\quad = \sum_{n=0}^{\infty} Ch_n^{(r)}(x \mid \lambda) \frac{t^n}{n!}.
\end{aligned} \tag{2.28}$$

Thus, by (2.28), we get

$$\begin{aligned}
&\frac{Ch_n^{(r)}(x \mid \lambda)}{n!} \\
&\quad = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{\lambda(x_1+\cdots+x_r)+x}{n} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \quad (n \geq 0).
\end{aligned} \tag{2.29}$$

Note that

$$\sum_{n=0}^{\infty} Ch_n^{(r)}(x \mid \lambda) \frac{t^n}{n!} = \left(\frac{2}{(1+t)^{\lambda}+1} \right)^r (1+t)^x = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} (x)_m Ch_{n-m}^{(r)}(\lambda) \binom{n}{m} \right) \frac{t^n}{n!}.$$

Thus, we get

$$Ch_n^{(r)}(x \mid \lambda) = \sum_{m=0}^{n} (x)_m Ch_{n-m}^{(r)}(\lambda) \binom{n}{m} \quad (n \geq 0). \tag{2.30}$$

By (2.28) and (2.29), we easily get

$$\begin{aligned}
Ch_n^{(r)}(x \mid \lambda) &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (\lambda x_1 + \cdots + \lambda x_r + x)_n \, d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
&= \sum_{l=0}^{n} S_1(n,l) \lambda^l \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(x_1 + \cdots + x_r + \frac{x}{\lambda} \right)^l d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
&= \sum_{l=0}^{n} S_1(n,l) \lambda^l E_l^{(r)}\left(\frac{x}{\lambda} \right).
\end{aligned} \tag{2.31}$$

Now, we consider the λ-Changhee and Korobov mixed-type polynomials which are given by the multivariate p-adic integral on $\mathbb{Z}_p$ as follows:

$$\begin{aligned}
&CK_n^{(r,s)}(x \mid \lambda) \\
&\quad = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} Ch_n^{(r)}(\lambda x_1 + \cdots + \lambda x_s + x \mid \lambda) \, d\mu_0(x_1) \cdots d\mu_0(x_s) \\
&\quad = \sum_{m=0}^{n} Ch_{n-m}^{(r)}(\lambda) \binom{n}{m} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (\lambda x_1 + \cdots + \lambda x_s + x)_m \, d\mu_0(x_1) \cdots d\mu_0(x_s),
\end{aligned} \tag{2.32}$$

where $r, s \in \mathbb{N}$.

Now, we observe that

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{\lambda x_1+\cdots+\lambda x_s+x}\,d\mu_0(x_1)\cdots d\mu_0(x_s)\\
&\quad=\left(\frac{\lambda t}{(1+t)^\lambda-1}\right)^s\left(\frac{\log(1+t)}{t}\right)^s(1+t)^x\\
&\quad=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}K_l^{(s)}(x\mid\lambda)D_{n-l}^{(s)}\binom{n}{l}\right)\frac{t^n}{n!}.
\end{aligned}\tag{2.33}$$

By (2.33), we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(\lambda x_1+\cdots+\lambda x_s+x)_n\,d\mu_0(x_1)\cdots d\mu_0(x_s)\\
&\quad=\sum_{l=0}^{n}K_l^{(s)}(x\mid\lambda)D_{n-l}^{(s)}\binom{n}{l}.
\end{aligned}\tag{2.34}$$

From (2.32) and (2.34), we have

$$CK_n^{(r,s)}(x\mid\lambda)=\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{m}{l}\binom{n}{m}Ch_{n-m}^{(r)}(\lambda)K_l^{(s)}(x\mid\lambda)D_{m-l}^{(s)}.\tag{2.35}$$

The generating function of $CK_n^{(r,s)}(x\mid\lambda)$ is given by

$$\begin{aligned}
&\sum_{n=0}^{\infty}CK_n^{(r,s)}(x\mid\lambda)\frac{t^n}{n!}\\
&\quad=\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}Ch_n^{(r)}(\lambda x_1+\cdots+\lambda x_s+x\mid\lambda)\,d\mu_0(x_1)\cdots d\mu_0(x_s)\frac{t^n}{n!}\\
&\quad=\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{-times}}\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{s\text{-times}}(1+t)^{\lambda y_1+\cdots+\lambda y_r+\lambda x_1+\cdots+\lambda x_s+x}\\
&\qquad\times d\mu_{-1}(y_1)\cdots d\mu_{-1}(y_r)\,d\mu_0(x_1)\cdots d\mu(x_s)\\
&\quad=\left(\frac{2}{(1+t)^\lambda+1}\right)^r\left(\frac{\lambda\log(1+t)}{(1+t)^\lambda-1}\right)^s(1+t)^x.
\end{aligned}\tag{2.36}$$

Theorem 2.12 *For $r,s\in\mathbb{N}$ and $n\geq 0$, we have*

$$CK_n^{(r,s)}(x\mid\lambda)=\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{m}{l}\binom{n}{m}Ch_{n-m}^{(r)}(\lambda)K_l^{(s)}(x\mid\lambda)D_{m-l}^{(s)}.$$

We consider the Korobov and λ-Changhee mixed-type polynomials, which are given by

$$KC_n^{(r,s)}(x\mid\lambda)=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}K_n^{(r)}(\lambda x_1+\cdots+\lambda x_s+x\mid\lambda)\,d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_s),\tag{2.37}$$

where $r,s\in\mathbb{N}$ and $n\geq 0$.

Then, by (2.37), we get

$$\begin{aligned} &KC_n^{(r,s)}(x \mid \lambda) \\ &= \sum_{m=0}^{n} \binom{n}{m} K_m^{(r)}(\lambda) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (\lambda x_1 + \cdots + \lambda x_s + x)_{n-m} \, d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_s) \\ &= \sum_{m=0}^{n} \binom{n}{m} K_m^{(r)}(\lambda) Ch_{n-m}^{(s)}(x \mid \lambda). \end{aligned} \tag{2.38}$$

The generating function of $KC_n^{(r,s)}(x \mid \lambda)$ is given by

$$\begin{aligned} &\sum_{n=0}^{\infty} KC_n^{(r,s)}(x \mid \lambda) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} K_n^{(r)}(\lambda x_1 + \cdots + \lambda x_s + x \mid \lambda) \, d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_s) \frac{t^n}{n!} \\ &= \left(\frac{\lambda t}{(1+t)^{\lambda} - 1} \right)^r \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{\lambda x_1 + \cdots + \lambda x_s + x} \, d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_s) \\ &= \left(\frac{\lambda t}{(1+t)^{\lambda} - 1} \right)^r \left(\frac{2}{(1+t)^{\lambda} + 1} \right)^s (1+t)^x. \end{aligned} \tag{2.39}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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